**Methods of Applied Math I** **Qualifying Exam Fall 2018**

Solve ALL 6 problems. Notes, books or calculators are NOT allowed.
**Be detailed in your answer**.

1. (20 points) Show that the system
$$\dot{x} = -y + x(1 - 2x^2 - 3y^2)\ , \qquad \dot{y} = x + y(1 - 2x^2 - 3y^2)$$
has at least one closed path in the phase plane.

2. (15 points) Consider the nonlinear dynamical system
$$\dot{x} = x\left(1 - \frac{x}{2} - y\right), \qquad \dot{y} = y\left(x - 1 - \frac{y}{2}\right)$$
i) Find all fixed points.
ii) Determine whether each fixed point is linearly stable or not and draw its local phase portrait.

3. (15 points) Consider the ODE (here, $0 < \varepsilon \ll 1$)
$$\frac{d^2y}{dt^2} + \omega_0^2 y = \varepsilon\left[\frac{dy}{dt} + \left(\frac{dy}{dt}\right)^2\right] .$$
Define $t_0 = t\ ,\ \ t_1 = \varepsilon t$ and let $y = y_0(t_0, t_1) + \varepsilon y_1(t_0, t_1) + \cdots$ .
   (a) Write down the $O(1)$ and $O(\varepsilon)$ equations for $y_0$ and $y_1$.
   (b) Solve the $O(1)$ equation.
   (c) Identify the secular term(s) and write down the condition(s) to guarantee their removal.

4. (15 points) Consider the dynamical system
$$\dot{x} = \left(r - x^2\right)\left(r - 2 + x^2\right),$$
where $r$ is a real parameter. Find all the fixed point(s). Discuss the linear stability analysis of these fixed point(s) and explain their dependence on $r$. How many bifurcation point(s) are there?

5. (20 points) Consider the boundary value problem
$$\varepsilon^2 \frac{d^2u}{dx^2} + (2x+1)\frac{du}{dx} + 2u = 0\ , \qquad u(0) = 1\ ,\ \ u(1) = 2\ ,$$
where $0 < \varepsilon \ll 1$. Find one term of the inner and outer expansions and match them.

6. (15 points) (i) Find three term expansion in terms of $0 < \varepsilon \ll 1$ for
$$x^2 - 2\varepsilon x - \varepsilon = 0.$$
(ii) Find two term approximation to the real roots of the equation
$$\varepsilon x^3 = x + 1.$$